

From Coarse-Graining to Holography in Loop Quantum Gravity

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We discuss the relation between coarse-graining and the holographic principle in the framework of loop quantum gravity and ask the following question: when we coarse-grain arbitrary spin network states of quantum geometry, are we integrating out physical degrees of freedom or gauge degrees of freedom? Focusing on how bulk spin network states for bounded regions of space are projected onto boundary states, we show that all possible boundary states can be recovered from bulk spin networks with a single vertex in the bulk and a single internal loop attached to it. This partial reconstruction of the bulk from the boundary leads us to the idea of realizing the Hamiltonian constraints at the quantum level as a gauge equivalence reducing arbitrary spin network states to one-loop bulk states. This proposal of “dynamics through coarse-graining” would lead to a one-to-one map between equivalence classes of physical states under gauge transformations and boundary states, thus defining holographic dynamics for loop quantum gravity.

Loop quantum gravity sets up a non-perturbative framework for quantum gravity, with evolving quantum state of geometries and area and volume operators with quantized spectra at the Planck scale (for reviews of both the basic formalism and recent developments, see [1–4]). It faces a triptych of interlaced issues: the coarse-graining of quantum geometry states from the Planck scale to larger scales, the definition of quantum dynamics consistent with the holographic principle and the implementation of (discretized) diffeomorphism at quantum level as the fundamental gauge symmetry of the theory (or, in other words, the implementation of a relativity principle for quantum geometry). These encompass more technical questions, such as anomaly cancellation, a well-behaved continuum limit and the perturbative renormalisation of quantum gravity corrections. In this short letter, we would like to discuss the relation between coarse-graining and holography.

Let us describe the standard setting for coarse-graining LQG. Quantum states of geometry are spin networks, that is graphs dressed with algebraic data: $SU(2)$ representations, or spins, on the graph links and singlet states, or intertwiners, on the graph nodes. These spin networks define the bulk geometry, with the underlying graph representing a network of points as the backbone of the 3d space. A node represents an elementary chunk of volume and is usually thought of geometrically in terms of a dual surface surrounding it. We now split up space into regions by partitioning the spin network into connected collections of nodes, as shown on fig.1. The procedure is to coarse-grain each region to a single node, thus leading to a coarse-grained network describing the coarse-grained geometry of space. The data attached to each coarse-grained node should reflect the information available on the geometry of the corresponding region of space: we define a projector mapping quantum states of bulk geometry inside each region onto states living on

the region’s boundary surface. This boundary surface is thought of as the dual surface to the coarse-grained node and the projected boundary state is the new algebraic data attached to coarse-grained node.

Actually this is exactly the same framework as when discussing the implementation of the holographic principle in loop quantum gravity by describing the bulk geometry through the dynamics of holographic screens throughout space, except for an apparent essential discrepancy in the point of view: coarse-graining means naturally loss of information, while the holographic principle claims that all the physically relevant degrees of freedom of the bulk are described by a surface theory without loss of information. This is resolved by underlining a crucial difference in the premises: kinematics versus dynamics.

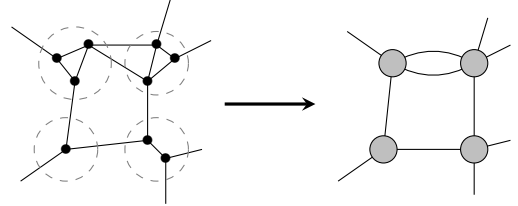


FIG. 1: A two-dimensional illustration of the partitioning of the graph in order to coarse-grain the spin network state.

Indeed, the coarse-graining procedure is to be applied to arbitrary quantum spin network states, not necessarily satisfying the Hamiltonian constraints: the goal is to understand the large scale structure emerging from arbitrary Planck scale quantum geometries. On the other hand, the holographic principle is a statement about the quantum gravity dynamics and applies to physical states satisfying the Hamiltonian constraints. This leads to one big question: is the information, lost when coarse-graining by projecting the bulk geometry onto the boundary, physically-relevant or is it pure gauge (for instance, under diffeomorphisms generated by the quantum Hamiltonian constraints)? For example¹, in topological BF theory, all the detail of the graph within a region can be

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gauged out and physical spin network states exactly projected onto the topological defects living on its boundary.

Actually, this natural interplay between holography and topological invariance was at the root of the proposal of considering general relativity as a constrained BF theory and realizing quantum gravity from topological quantum field theory (TQFT), which materialized into the spinfoam path integral for loop quantum gravity [8–11].

This logic leads us to a drastic proposal to implement the dynamics of loop quantum gravity: we could consider all the information lost in the coarse-graining projection are gauge degrees of freedom and take this as a definition of the Hamiltonian constraints defining the diffeomorphism invariance at the quantum level. This can be understood as an extension of the topological invariance (in particular, of the triangulation invariance under Pachner moves) of BF theories to theories, such as (quantum) gravity, which are non-topological but nevertheless holographic. This would be in spirit similar to the proposal of “dynamics through coarse-graining” by Dittrich and Steinhaus [12] and sets coarse-graining procedures for spin network states as a fundamental building block of the theory.

Following this line of thought, we propose to analyze the projection from bulk geometry to boundary state in loop quantum gravity and focus on two points. First, can we classify² all the bulk spin network states (graph and algebraic data) leading to a same boundary state? What is the “redundant” information stored in the bulk?

Second, what is the simplest bulk structure compatible with a given boundary state? We investigate these questions in the most straightforward LQG formulation, with quantum states of geometry defined as SU(2) spin networks and not as some notion of extended spin network³.

We consider a bounded region of a spin network state,

with the graph puncturing the boundary surface at N points (see e.g. [20] for a discussion on the definition of quantum surfaces in loop quantum gravity). Each puncture carries an arbitrary spin, so that the boundary state lives in the tensor product of N arbitrary spins with the only constraint that the sum of all N boundary spins is an integer:

$$\mathcal{B}_N = \bigoplus_{\sum_{i=1}^N j_i \in \mathbb{N}} \mathcal{B}_{j_1, \dots, j_N} \subset \left[\bigoplus_{j \in \frac{\mathbb{N}}{2}} \mathcal{V}_j \right]^{\otimes N}, \quad (1)$$

$$\mathcal{B}_{j_1, \dots, j_N} = \bigotimes_{i=1}^N \mathcal{V}_{j_i}, \quad \dim \mathcal{B}_{j_1, \dots, j_N} = \prod_i d_{j_i},$$

where j_i are the spins carried by the N links puncturing the boundary surface and the dimension of the spin- j representation $d_j = \dim \mathcal{V}_j = 2j + 1$. If the spin network graph in the bulk (here referring to the interior of the bounded region) is a tree, i.e. does not contain any loop, then it is known that the resulting boundary states are necessarily intertwiners, that is singlet states with vanishing overall spin. This is clearly does not allow for arbitrary boundary state and can not be the generic case. So the question we will address in this letter is how much should we complicate the graph in order to allow for all possible boundary states.

I. BOUNDARY STATES FOR A SINGLE LOOP IN THE BULK

Let us start by the simplest extension of a tree graph and consider a graph with a single loop. We will show that we get all possible boundary states if we allow for an arbitrary SU(2) holonomy around the loop.

A useful decomposition of the boundary Hilbert space is to partition it in terms of the closure defect [6, 7, 20, 21]. Mathematically, this means recoupling all the spins j_i carried by the N punctures into a single overall spin s . This spin, living on the intermediate channel, is necessarily an integer⁴:

$$\mathcal{B}_N = \bigoplus_{s \in \mathbb{N}} \mathcal{B}_N^s, \quad \mathcal{B}_N^s = \bigoplus_{\{j_i\}_{i=1 \dots N}} \mathcal{B}_{j_1, \dots, j_N}^s, \quad (2)$$

$$\mathcal{B}_{j_1, \dots, j_N}^s = \mathcal{V}_s \otimes \text{Inv} \left[\mathcal{V}_s \otimes \bigotimes_{i=1}^N \mathcal{V}_{j_i} \right]. \quad (3)$$

¹ Another example is provided by the distinction between bulk entropy and boundary entropy for black holes in loop quantum gravity [5]: holonomies wrapping around internal loops of the spin network graph within a region are non-trivial degrees of freedom generating entropy. If we want the bulk entropy to scale with the area at leading order, we should either bound the number of loops that can develop in the bulk or consider that these internal holonomies are partly gauge degrees of freedom. On the other hand, it was shown recently in [6, 7] that, as soon as the number of internal loops is large enough, the boundary state is automatically thermal and respects the area-entropy law.

² We could go further by formulating this question as a Kadison-Singer problem. Given a diagonal boundary state, attributing values to a maximal set of commuting observables probing the geometry of the boundary, what are its possible extension of a quantum state of the bulk geometry? This could be especially interesting from the perspective of the relation between this question and the sparsification of networks, which is relevant to the issue of coarse-graining spin networks.

³ Various extensions of spin networks have been introduced since the original formulation of loop quantum gravity in terms of SU(2) spin networks: SL(2, \mathbb{C}) simple and projected spin networks used in spinfoam path integrals [13, 14], spin networks labeled with representations of the Drinfeld double $\mathcal{D}(\text{SU}(2))$ to

account for curvature and torsion excitations [15, 16], double spin networks with holonomies both along the graph edge and looping around them [17] or the recently developed “loop gravity string” framework mixing the SU(2) algebraic structures with the Virasoro algebra of surface diffeomorphism living on each boundary surface [18, 19].

As depicted on fig.2, a convenient basis of each subspace \mathcal{B}_N^s is given by choosing a basis state $|s, M\rangle$ with magnetic moment M in \mathcal{V}_s and an intertwiner \mathcal{I} recoupling the N spins j_i with the spin s :

$$\mathcal{I} \in \text{Inv} \left[\mathcal{V}_s \otimes \bigotimes_{i=1}^N \mathcal{V}_{j_i} \right], \quad \langle s, M | \mathcal{I} \rangle \in \mathcal{B}_{j_1, \dots, j_N}^s. \quad (4)$$

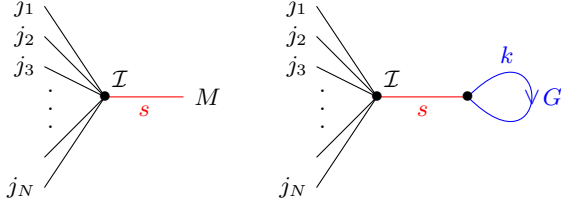


FIG. 2: On the left hand side, we illustrate the basis states of the boundary Hilbert space \mathcal{B}_N , as given in eqn.(2-4), labeled by the boundary spins j_1, \dots, j_N , the closure defect spin s and magnetic moment M and intertwiner \mathcal{I} recoupling the boundary spins and the closure defect. On the right hand side, we draw spin network states for a one-loop bulk, as defined in eqn.(5), with the intermediate spin s on the link between the boundary spins j_1, \dots, j_N and the internal loop carrying the spin k and the holonomy $G \in \text{SU}(2)$.

Now let us look at spin network states with boundary spins j_i based on the single node graph with a self-loop, as drawn on fig.2. A convenient basis is to separate the loop from the boundary edges, choose a spin k carried by the loop and once more intertwine the boundary spins j_i into a single overall spin s . Then the spin network states is defined by two intertwiners, one defining the recoupling of the j_i 's to s and one defining the recoupling of two copies of the spin k into s . The latter is actually unique once k and s are given since it is a 3-valent node. The boundary states defined by these spin network basis states are expressed in terms of the holonomy $G \in \text{SU}(2)$ carried by the loop and the intertwiner $\mathcal{I} \in \mathcal{B}_{j_1, \dots, j_N}^s$:

$$\psi_{\{j_i\}}^{s, k, \mathcal{I}}[G] = D_{m, \tilde{m}}^k(G) C_{\tilde{m}, M | m}^{k, s | k} \langle s, M | \mathcal{I} \rangle \in \mathcal{B}_{j_1, \dots, j_N}^s. \quad (5)$$

with the Wigner-matrix of the holonomy,

$$D_{m, \tilde{m}}^k(G) = \langle k, \tilde{m} | G | k, m \rangle,$$

⁴ A bounded region \mathcal{R} is a set of nodes of the spin networks, with all the links connected to them. We distinguish the interior links, whose both ends belong to \mathcal{R} and the boundary links with a single node in \mathcal{R} . The parity condition at a node v is that the sum of the spins around that node is an integer. Summing all the parity conditions for the vertices in \mathcal{R} implies that the sum of the boundary spins is necessarily an integer. This holds as long as there is no source of torsion (e.g. fermions) within the region.

and the Clebsh-Gordan coefficient recoupling three spins:

$$C_{\tilde{m}, M | m}^{k, s | k} = \langle k, m | (k, \tilde{m}) \otimes (s, M) \rangle.$$

We would like to prove that these states $\psi_{\{j_i\}}^{s, \mathcal{I}}[G]$ cover the whole boundary space $\mathcal{B}_{j_1, \dots, j_N}$ for any given boundary spins j_1, \dots, j_N . We compute⁵ the density matrix ρ_k integrating over the holonomy G carried by the loop at fixed loop spin k :

$$\begin{aligned} \rho_k &= \int dG |\psi_{\{j_i\}}^{s, k, \mathcal{I}}[G]\rangle \langle \psi_{\{j_i\}}^{s, k, \mathcal{I}}[G]| \\ &= \int dh \frac{\chi_k(h)^2}{2k+1} D_{MM'}^s(h) \langle s, M | \mathcal{I} \rangle \langle \mathcal{I} | s, M' \rangle, \end{aligned} \quad (6)$$

where $\chi_k(h) = \text{Tr } D^k(h)$ is the character of the group element $h \in \text{SU}(2)$ for the spin- k representation. An interesting formula⁶ for distributions over $\text{SU}(2)$ is:

$$\delta_{\text{SO}(3)}(h) = \sum_{n \in \mathbb{N}} (2n+1) \chi_n(h) = -2 \sum_{k \in \frac{\mathbb{N}}{2}} \chi_k^2(h). \quad (7)$$

Since the spin s in the intermediate channel is always an integer, we can sum over the loop spin k and obtain the density matrix ρ :

$$\rho = - \sum_k \frac{2k+1}{2} \rho_k = \langle s, M | \mathcal{I} \rangle \langle \mathcal{I} | s, M \rangle = \mathbb{I}_{\mathcal{B}_{j_1, \dots, j_N}^s}, \quad (8)$$

which we recognize as the identity of the boundary Hilbert space $\mathcal{B}_{j_1, \dots, j_N}^s$.

This not only shows that the boundary states induced by a one-loop bulk cover the whole boundary Hilbert space, but it also provides an explicit decomposition of the identity on the boundary space in terms of intertwiners with one self-loop. This allows to decompose an arbitrary boundary state in \mathcal{B}_N in terms of those one-loop spin networks, thus providing them with an interpretation of coherent bulk states.

⁵ We use the useful realization of the product of two Clebsh-Gordan coefficients as an integral over the product of three Wigner-matrices:

$$C_{\tilde{m}, M | m}^{k, s | k} \overline{C_{m' | \tilde{m}', M'}^{k | k, s}} = \int dh D_{MM'}^s(h) D_{\tilde{m} \tilde{m}'}^k(h) D_{m' m}^k(h^{-1}).$$

⁶ We distinguish the δ -distribution on $\text{SU}(2)$, which expands over all spins in $\mathbb{N}/2$, and the δ -distribution on $\text{SO}(3)$, which involves only integer spins. Then we use the tensor product formula for characters, recoupling two copies of a spin k :

$$\forall k \in \frac{\mathbb{N}}{2}, \quad \chi_k^2 = \sum_{n=0}^{2k} \chi_n, \quad \forall n \in \mathbb{N}^*, \quad \chi_n = \chi_{\frac{n}{2}}^2 - \chi_{\frac{n-1}{2}}^2.$$

II. FROM BOUNDARY STATES TO BULK STATES AND VICE-VERSA

The mathematical result proved above has a direct application to the question of the local reconstruction of the bulk geometry from the boundary state in LQG. The minimal reconstruction does not require a complex graph structure in the bulk, we only need a single loop in the bulk. Using the formula for the decomposition of the identity on $\mathcal{B}_{j_1, \dots, j_N}^s$ in terms of one-loop bulk states, an arbitrary boundary state unfolds on a graph with one internal loop and determines the (probability amplitude for the) values of the internal holonomy G , internal spin k living on the loop and the closure defect spin s . From this point of view, the boundary state does not carry further information on the bulk structure and it seems that we should consider any more complicated graph structure in the bulk as gauge degrees of freedom.

This can be put in contrast with the loopy spin network framework developed in [22] in an attempt to define a consistent coarse-graining procedure for loop quantum gravity, proposing to consider the dynamics of spin networks on a fixed graph background but allowing for an arbitrary number of self-loops attached to each node. These self-loops (or little loops as named in [22]) are interpreted as representing local excitations of curvature (localized at each node of the background graph). Here, our result seems to collapse this multi-loop structure and require a single loop at each graph node to deal with the coarse-graining of the theory.

So we formulate a very crude proposal, which will necessarily need to be refined: the Hamiltonian constraints imposing diffeomorphism invariance can be implemented in loop quantum gravity as a gauge invariance rendering all bulk graph and spin network states to be gauge-equivalent to a one-loop bulk state. Although this would drastically simplify the theory, it does not reduce to a topological BF theory and is clearly an extension beyond it: in BF theory, all internal loops are pure gauge and can be entirely gauged-out, while here we allow for one-loop states and thus for local curvature and local degrees of freedom living on those loops. This is consistent with the formulation of general relativity as a BF theory with extra constraints, which is the standard starting point for constructing spinfoam path integrals for implementing the dynamics of loop quantum gravity [9, 23–25] (see [26] for a review).

In order to make this proposal explicit, we need the operator(s) at the quantum level which generate(s) this gauge invariance (under diffeomorphism) and map(s) any bulk spin network state onto the corresponding one-loop bulk state. Here we follow the coarse-graining procedure, “coarse-graining by gauge-fixing” defined in detail in [21, 22, 27] to project any bulk state onto the corresponding boundary state and then use the formula presented in the previous section to map that boundary state onto the corresponding one-loop bulk state. More generally, we can generalize this logic to any coarse-graining procedure

defined for kinematical spin network states and turn it into a definition of the gauge invariance implementing the dynamics of spin networks in loop quantum gravity. This underlines the fundamental physical relevance of a coarse-graining procedure for spin networks.

Finally, this proposal would automatically implement the holographic principle in loop quantum gravity: physical states for the bulk geometry would be exactly gauge-equivalent to one-loop bulk states and thus, in other words, gauge-equivalent to boundary states.

Outlook & Conclusion

We proved that it is possible to interpret all boundary states, on a quantum surface with N punctures, as spin network states on a bulk graph consisting in a unique vertex dressed with an intertwiner and a single little loop attached to it and carrying an arbitrary $SU(2)$ holonomy. This led us to the idea of “dynamics by coarse-graining” for loop quantum gravity, where the Hamiltonian constraints would translate to gauge transformations mapping arbitrary bulk spin network states to one-loop bulk states. This would mean that physical states (solving the Hamiltonian constraints) would be in a one-to-one correspondence with boundary states. It would be a straightforward extension of the dynamics of BF theory: (little) loops would not be completely pure gauge but almost, so that we gauged-out all internal loops but one for any bounded region of space. Although this is, for now, more the outline of a proposal than an explicit realization, this framework would automatically implement the holographic principle in the heart of loop quantum gravity.

To establish this line of investigation, the next step would be to check whether our conclusion is robust to the proposed extensions of spin network states, for instance if we consider both curvature and torsion defects with spin networks labeled with Drinfeld double representations (based on exponentiated fluxes) as in [15, 28] or if we add the information of the Virasoro currents living on the surfaces (encoding information on the intrinsic geometry of the surface) as advocated in [19]. Considering the latter proposal, it will be very interesting to understand how the coarse-graining procedure for spin networks could lead to conformal field theory states on the boundary and how such a proposal for the loop quantum gravity dynamics fits with the possibility of a local gravity/CFT correspondence.

Finally, it would be essential to understand the relation between the point of view we developed and the recent result by Anzà and Chirco [7] that a complicated-enough bulk (defined by a graph with a large enough number of internal loops) would typically lead to thermal boundary states. This points towards the necessity of a finer analysis of the bulk-to-boundary relation in loop quantum gravity: considering an arbitrary *mixed boundary state* defined by an arbitrary density matrix, can we identify it

the projection of a *pure bulk state* defined on a potentially complicated bulk graph?

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